# Magnetostatic Surface-Wave Propagation in Ferrite Thin Films with Arbitrary Variations of the Magnetization Through the Film Thickness

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***Abstract*** **—A variational formulation for the magnetostatic problem in an anisotropic and inhomogeneous region bounded by perfect conductors is described. The method is applied to the special case of magnetostatic surface-wave (MSSW) modes propagating in a ferrite thin film with arbitrary variations of the saturation magnetization through the film thickness. Methods for calculating dispersion relations, delay characteristics, and magnetostatic potential functions are discussed. The functional that is minimized is interpreted in terms of contributions to the mode energy. Also, concepts pertaining to homogeneous films such as mode bandwidth and dimensional scaling effects are extended to the general inhomogeneous case. Calculations for a two-layer film with a gradual transition region and an ion-implanted film are presented as numerical examples.**

## I. Introduction

MAGNETOSTATIC WAVES propagating in a ferrite slab or thin-film magnetized in-plane were first described by Damon and Eshbach [1]. These waves may lead to a new class of microwave devices such as tunable delay lines, filters, and resonators [2].

An undesirable characteristic for many applications, however, is the highly dispersive nature of the waves. Experimentally, as well as theoretically, it has been shown that the delay of magnetostatic waves can be controlled by using multilayer ferrite-dielectric structures above a ground plane [3]–[17]. Evidence also exists that inhomogeneities in the magnetization and/or the bias field can be used to control the dispersion as well as guide and localize the magnetostatic mode energy [18]–[26].

Multilayer structures can be viewed as special cases of an arbitrary thickness variation of the magnetization $M_s$. Thickness variations in $M_s$ have also been induced by ion implantation [27], [28], and occur naturally in ferrite thin films at the transition layer between the ferrite and the nonmagnetic substrate [29].

The problem of arbitrary inhomogeneities cannot be easily attacked by classical boundary value techniques. Consequently, methods based upon variational principles have been introduced for analyzing nonuniform geometries [23], [30], [31]. Here we describe a technique for analyzing magnetostatic surface waves (MSSW's) propagating in a thin film with arbitrary variations in the magnetization through the film thickness. The technique is based on a variational principle [30] and a slight modification of Ritz' method. We expand the potential in the ferrimagnetic region in a complete set of functions, and then, using the variational principle, reduce the problem to an infinite linear system in the coefficients of the expansion. In principle, the solution can be obtained to any degree of accuracy by increasing the number of terms kept in the potential expansion. As numerical examples, results for a two-layer film with a gradual transition region and an ion-implanted film are presented.

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## II. Equivalence of Variational and Boundary Value Methods

Consider a region $V$ of space, bounded by the closed surface $S$. Suppose that a potential field $\psi(\bar{r})$ exists in this region and satisfies the equation

$$\nabla\cdot(\bar{\bar{\mu}}\cdot\nabla\psi)=0 \tag{1}$$

with

$$\hat{n}\cdot(\bar{\bar{\mu}}\cdot\nabla\psi)|_{S_1}=0 \tag{2a}$$

$$\psi(\bar{r})|_{S_2}=g_s(\bar{r}) \tag{2b}$$

and $S_1+S_2=S$. Here $\bar{\bar{\mu}}$ is the permeability tensor and $g_s$ is a given function on $S_2$. The permeability tensor $\bar{\bar{\mu}}$ is a function of position, in general. Equations (2a) and (2b) form a set of mixed boundary conditions that guarantee the uniqueness of the field solution.

Given the potential function $\psi$ satisfying (1) and (2), the magnetostatic vector fields can be obtained from

$$\bar{h}=-\nabla\psi \tag{3}$$

$$\bar{b}=\bar{\bar{\mu}}\cdot\bar{h}. \tag{4}$$

Equation (1) is an ordinary differential equation of the second degree which, as is well known, can always be the Euler–Lagrange equation of an appropriate variational problem [32]. Appendix A shows that the above magnetostatic problem is equivalent to requiring the first variation

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of the following functional to vanish:

$$W=\int_V L\,dV \tag{5}$$

where

$$L=\mu_{ij}\psi_j\psi_i^* \tag{6}$$

$$\psi_k=\frac{\partial\psi}{\partial x^k},\qquad k=1,2,3. \tag{7}$$

The * indicates the complex conjugate, and summation over repeated indices is implied unless otherwise stated. Further, integration by parts shows the stationary value of $W$ to be zero (Appendix B).

## III. Application to Magnetostatic Surface Waves

An effective method for solving variational problems is that of Ritz, where an expansion of the unknown field in complete, orthogonal functions is substituted into the functional to be extremized. This gives a system of equations in the coefficients of the expansion. Solving for these coefficients then determines the field.

Here we apply the above method to the propagation of magnetostatic surface waves in infinite ferrimagnetic slabs between ground planes. Let us assume a ferrimagnetic slab of thickness $2s$ placed between parallel infinite perfect conductors as in Fig. 1. Consider a coordinate system with origin at the middle of the slab with the $z$-axis parallel to the slab and the conductors. Due to the symmetry of the problem, the analysis becomes two-dimensional. We also assume a bias field $\overline{H}=H_0\hat{z}$ and a small-signal time dependence of the form $\exp(-i\omega t)$. The permeability tensor of the ferromagnetic material takes the form

$$\bar{\bar{\mu}}=\mu_0\begin{bmatrix}1+\chi & -i\kappa & 0\\ i\kappa & 1+\chi & 0\\ 0 & 0 & 1\end{bmatrix} \tag{8}$$

where

$$1+\chi=\frac{\omega_0(\omega_0+\omega_m)-\omega^2}{\omega_0^2-\omega^2} \tag{9}$$

$$\kappa=\frac{\omega_m\omega}{\omega_0^2-\omega^2} \tag{10}$$

$\omega_0=-\gamma\mu_0H_0$, $\omega_m=-\gamma\mu_0M_s(x)$, $\gamma$ is the gyromagnetic ratio (negative), and $M_s(x)$ is the saturation magnetization of the slab as a function of position.

Substituting (8) into (1) gives the general form of Walker's equation in the ferrite [19]

$$(1+\chi)\frac{\partial^2\psi}{\partial x^2}+\frac{\partial(1+\chi)}{\partial x}\frac{\partial\psi}{\partial x}+\frac{\partial^2\psi}{\partial y^2}=0. \tag{11}$$

In the dielectric, (11) reduces to Laplace's equation.

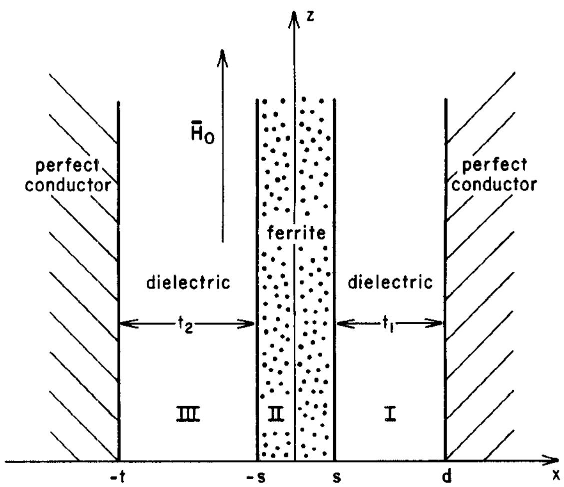


Fig. 1. Ferrite film geometry and coordinate system used.

We are interested in potentials that describe waves propagating along the $y$-axis. These potentials can be written as

$$\psi_{\mathrm{I}}=\left\{A_1e^{-\beta x}+A_2e^{\beta x}\right\}e^{i\nu\beta y} \tag{12a}$$

$$\psi_{\mathrm{II}}=C_nf_n(x)e^{i\nu\beta y} \tag{12b}$$

$$\psi_{\mathrm{III}}=\left\{D_1e^{-\beta x}+D_2e^{\beta x}\right\}e^{i\nu\beta y} \tag{12c}$$

where $\nu=\pm1$ is a directional parameter, $\beta$ is restricted to positive values, and the functions $f_i(x)$ form a complete set in the interval $[-s,s]$. Using (12) and the fact that the fields vanish in the region outside of the two ground planes, the functional (5) becomes

$$\begin{aligned}w=\mu_0\int_{-t}^{-s}&\left[\beta^2|D_1e^{-\beta x}-D_2e^{\beta x}|^2\right.\\ &\left.+\nu^2\beta^2|D_1e^{-\beta x}+D_2e^{\beta x}|^2\right]dx\\ &+\mu_0\int_{-s}^{s}\left\{\left[(1+\chi)C_nf_n'+\kappa\nu\beta C_nf_n\right]C_if_i'\right.\\ &\left.+\left[\kappa C_nf_n'+\nu\beta(1+\chi)C_nf_n\right]\nu\beta C_if_i\right\}dx\\ &+\mu_0\int_s^d\left[\beta^2|A_1e^{-\beta x}-A_2e^{\beta x}|^2\right.\\ &\left.+\nu^2\beta^2|A_1e^{-\beta x}+A_2e^{\beta x}|^2\right]dx\end{aligned} \tag{13}$$

where $w$ represents $W$ per unit area since, in our case, $L$ is a function of $x$ only.

To apply the variational principle to the functional, we need to have a continuous $\psi$ in (6); as for the boundary conditions (2), the problem can be described by homogeneous Neumann conditions on the ground planes. Continuous $\psi$ implies

$$A_1e^{-\beta s}+A_2e^{\beta s}=C_nf_n(s) \tag{14a}$$

$$D_1e^{\beta s}+D_2e^{-\beta s}=C_nf_n(-s). \tag{14b}$$

Requiring the normal $b$ field to vanish at the ground planes gives

$$A_1e^{-\beta d}-A_2e^{\beta d}=0 \tag{15a}$$

$$D_1e^{\beta t}-D_2e^{-\beta t}=0. \tag{15b}$$

Using the boundary conditions (14)–(15), we can eliminate $A_1$, $A_2$, $D_1$, and $D_2$ from (13) which, after performing the integrals over the dielectric regions, becomes

$$\begin{aligned} w = \mu_0 \beta C_i C_j \left[ th(\beta t_2) f_i f_j|_{-s} + th(\beta t_1) f_i f_j|_s \right] \\ + \mu_0 C_i C_j \int_{-s}^{s} \left[ (1+\chi)\left\{ f_i' f_j' + \beta^2 f_i f_j \right\} \right. \\ \left. + \kappa\nu\beta \left\{ f_i' f_j + f_i f_j' \right\} \right] dx. \end{aligned} \tag{16}$$

In the first section, we stated that, for the correct potential $\psi$, the functional (16) has a vanishing first variation. Necessary conditions for this are

$$\frac{\partial w}{\partial C_k} = 0 \tag{17}$$

for all $k$. Applying (17) to (16), we get the following infinite linear system in the expansion coefficients $C_i$:

$$2\mu_0 a_{ki} C_i = 0 \tag{18a}$$

for all $k$, where

$$\begin{aligned} a_{ki} = \beta \left[ th(\beta t_2) f_i f_k|_{-s} + th(\beta t_1) f_i f_k|_s \right] \\ + \int_{-s}^{s} \left[ (1+\chi)\left( f_i' f_k' + \beta^2 f_i f_k \right) + \kappa\nu\beta \left( f_i' f_k + f_i f_k' \right) \right] dx. \end{aligned} \tag{18b}$$

The system (18a) has the trivial solution unless

$$\det(a_{ij}) = 0. \tag{19}$$

Our method consists of varying beta in such a way that (19) is satisfied. This process is repeated for different frequencies to obtain the dispersion relation $\beta = \beta(\omega)$.

Having found the wavenumber $\beta$, the equations (18a) are solved for the expansion coefficients $C_i$. Finally, the boundary conditions (14)–(15) can be used to determine the constants $A_1$, $A_2$, $D_1$, and $D_2$ in terms of the $C_i$'s, completing the solution for the potential (12). The group delay can be calculated by a numerical derivative of the dispersion relation.

## IV. Comparison with Ritz' Method

Normal application of Ritz' method would assume a potential of the form

$$\psi(x, y) = \varphi_0(x) e^{i\nu\beta y} \tag{20a}$$

with

$$\varphi_0(x) = b_j F_j(x) \tag{20b}$$

for the entire region of $x$ between the two ground planes. An analogous procedure for the calculation of $w$ would give

$$\begin{aligned} w = \mu_0 b_i b_j \int_{-t}^{d} (1+\chi) \left[ \left( F_i' F_j' + \beta^2 F_i F_j \right) \right. \\ \left. + \kappa\nu\beta \left( F_i' F_j + F_i F_j' \right) \right] dx \end{aligned} \tag{21}$$

where now the set $\{\cdots, F_i(x), \cdots\}$ is complete in the interval $[-t, d]$, and $\chi$ and $\kappa$ vanish in the dielectric gaps. Such an analysis would require many more terms than (12b) since the series would be called upon to describe the discontinuities in the slope of the potential at the film edges.

On the other hand, our method combines three electromagnetic problems, two of which (in the dielectric regions) have known analytic general solutions. The boundary conditions on the continuity of the field (14)–(15) together with the fact that mixed boundary conditions uniquely determine the solution to Laplace's equation [33], guarantee that the three fields in (12) exist simultaneously. Thus, our method can be viewed as a judicious combination of both the boundary value technique and Ritz' method.

A quantitative demonstration of the above comparison can be done for the case of an isolated film $(t_1, t_2 \to \infty)$ where the wavenumber can be calculated exactly. We have used Legendre polynomials for reasons to be explained in Section VII. We studied the wavenumber of a 30-$\mu$m homogeneous film of $M_s = 140$ kA/m using two sets of functions. In the first case, the functions $f_i$ were Legendre polynomials orthogonal in the region $-15$ $\mu$m to $+15$ $\mu$m (the entire film thickness). Using only two terms in (12b), $\beta$ was found with an error of 0.0136 percent at 2.96 GHz. By using three terms, the error in $\beta$ was reduced to 0.0012 percent. In the second case, the functions $f_i$ were Legendre polynomials orthogonal in the region $-15$ $\mu$m to $+25$ $\mu$m; that is, the orthogonality region was extended by 33 percent beyond the film thickness at one side. This required the series to reproduce the discontinuity in the slope of the potential at the film edge, as discussed above. In this case, the use of twenty terms in (12b) resulted in an error in $\beta$ of 3.2 percent at 2.96 GHz.

## V. Interpretation of the Functional

Using the constitutive relation $\bar{b} = \mu_0(\bar{h} + \bar{m})$ and the linearized Landau–Lifshitz equation of motion, the effective Lagrangian density $L$ can be expressed [34], [35]

$$L = \bar{b}\cdot\bar{h}^* = 4\left[ \frac{\mu_0}{4}|\bar{h}|^2 + \frac{\mu_0}{4}\frac{H_0}{M_s}|\bar{m}|^2 - i\frac{\mu_0}{4}\frac{\omega}{\omega_m}\hat{z}\cdot(\bar{m}\times\bar{m}^*) \right]. \tag{22}$$

The first two terms in (22) represent the small-signal magnetostatic and Zeeman energy densities, respectively. The last term is always real and can be interpreted as a small-signal pseudo-kinetic energy density associated with the precession of the magnetization [36]. (Morgenthaler has also interpreted this quantity in terms of a quasi-particle number density [34].) Thus, the Lagrangian density consists of the difference between the potential and pseudo-kinetic energy densities of the mode. Since $W = \int \bar{b}\cdot\bar{h}^* \, dV = 0$, we conclude that the net potential and pseudo-kinetic energies of the mode are equal.

The total magnetic energy density in a dispersive medium is [37]

$$\begin{aligned} u_m &= \frac{1}{4}\bar{h}^* \cdot \frac{\partial(\omega\bar{\bar{\mu}})}{\partial\omega}\cdot\bar{h} \\ &= \frac{1}{4}\bar{b}\cdot\bar{h}^* + \frac{\omega}{4}\bar{h}^*\cdot\frac{\partial\bar{\bar{\mu}}}{\partial\omega}\cdot\bar{h}. \end{aligned} \tag{23}$$

Using (8), the last term in (23) can be expressed

$$\frac{\omega}{4}\bar{h}^*\cdot\frac{\partial\bar{\bar{\mu}}}{\partial\omega}\cdot\bar{h} = \frac{i\mu_0}{4}\frac{\omega}{\omega_m}\hat{z}\cdot(\bar{m}\times\bar{m}^*). \tag{24}$$

The first and second terms in (23) are sometimes called the pseudo-energy density [38] and the dispersion energy density [39], respectively. In plasmas and ionic crystals, the electric dispersion energy density is associated with the kinetic energy of the ions or charge carriers [39]. Similarly, (24) identifies the dispersion energy density with the pseudo-kinetic energy density associated with the magnetic precession.

However, the precession of the magnetization is a gyroscopic motion that cannot contribute to the total energy of the system (hence our choice of the term pseudo-kinetic). Combining (22)–(24) gives

$$u_m = \frac{\mu_0}{4}|\bar{h}|^2 + \frac{\mu_0 H_0}{4M_s}|\bar{m}|^2 \tag{25}$$

verifying that the gyroscopic terms do not contribute to the total mode energy density. (Here we have omitted the crystalline anisotropy energy density for simplicity.)

Fishman and Morgenthaler [35] have shown that the integral of the last term in (22) over the ferrite will give the total mode energy. In the present context, this follows immediately from $W = 0$ and the equality of potential and pseudo-kinetic energies.

## VI. General Properties

The frequency range for magnetostatic surface waves in an homogeneous magnetization is given by [1]

$$\omega_L \le \omega \le \omega_H \tag{26a}$$

where

$$\omega_L = [\omega_0(\omega_0 + \omega_m)]^{1/2} \tag{26b}$$

and

$$\omega_H = \omega_0 + \omega_m/2. \tag{26c}$$

In this region, $(1+\chi)$ is always positive.

With the variational technique, we are able to investigate the general case where $M_s = M_s(x)$. Thus, (9) shows that $(1+\chi)$ is a function of both $x$ and $\omega$. We consider the quantity

$$\frac{\partial^2 L}{\partial \varphi' \partial \varphi'} \tag{27}$$

with $\varphi' = \partial\varphi/\partial x$ and $\varphi(x) = c_n f_n(x)$. For surface waves

$$\frac{\partial^2 L}{\partial \varphi' \partial \varphi'} = 1 + \chi. \tag{28}$$

For a uniform magnetization, (9) shows that $1+\chi$ is positive for all $\omega > \omega_L$. (In the dielectric regions, $1+\chi$ is clearly positive since $\chi$ vanishes.) The condition

$$\frac{\partial^2 L}{\partial \varphi' \partial \varphi'} > 0 \tag{29}$$

first investigated by A. M. Legendre [40], together with the vanishing of the first variation of $w$ in (5), shows that $w$ possesses a minimum for the correct field.

For nonuniform magnetization, we define the frequency (cf. (26b))

$$\omega_{L\,\max} = \left(\omega_0(\omega_0 + \max \omega_m)\right)^{1/2} \tag{30}$$

where $\max \omega_m$ is the value of $\omega_m$ corresponding to the maximum of $M_s(x)$. Thus, $1+\chi$ is positive definite for frequencies above $\omega_{L\,\max}$. We have always found a surface-wave mode having a low cutoff frequency of $\omega_{L\,\max}$. This is a confirmation and generalization of the result reported by Adkins and Glass [12]. For frequencies below $\omega_{L\,\max}$, the sign of the quadratic form (27) depends on the position. Although we have found modes in this frequency range, we have not been able to obtain satisfactory convergence. Consequently, these modes are not discussed in the present paper. This is the condition for which virtual surface modes may exist [20] and needs more investigation.

It is well known that the quantity $\beta s$ is invariant under a change in film thickness for uniform films without a ground plane. It can be shown (Appendix C) that this result is also valid for inhomogeneous films with ground planes, if $\chi$ and $\kappa$ transform as scalars under the point transformation

$$\tilde{x} = \epsilon x, \qquad \epsilon > 0. \tag{31}$$

That is, if $\tilde{\beta}$ is the wavenumber of the MSSW of an expanded $(\epsilon > 1)$ or contracted $(\epsilon < 1)$ film geometry, then

$$\beta(\omega) \equiv \epsilon\tilde{\beta}(\omega). \tag{32}$$

Taking the derivative of (32) with respect to $\omega$ gives the corresponding result for the group delay

$$\tau(\omega) \equiv \epsilon\tilde{\tau}(\omega) \tag{33}$$

where $\tilde{\tau}$ is the delay of the MSSW in the transformed geometry. As discussed in Appendix C, the mode passbands of the two geometries are identical.

## VII. Numerical Examples

Legendre Polynomials defined in the interval $[-s, s]$ have been chosen for the basis functions $\{\cdots, f_i(x), \cdots\}$ in our analysis. These functions are convenient because of their orthogonality in the interval $[-s, s]$ and the fact that a good approximation to the potential function of an isolated, uniform film can be obtained using only $f_0$ and $f_1$ (see also Section IV). A computer program has been written to calculate the matrix (18b) and its determinant. The dispersion relation, the potential, and the delay characteristics of the waves are obtained as described in Section III.

As an example, consider the two-layer film with a gradual transition region shown in Fig. 2. The potentials for waves traveling in both the positive $(\nu = +1)$ and negative $(\nu = -1)$ $y$ directions are also shown. The corresponding delays are plotted in Fig. 3. The maximum value of $M_s$ is 143 kA/m (1797 G) while the minimum is 110 kA/m (1383 G) occurring at $x = +s$ and $x = -s$, respectively.

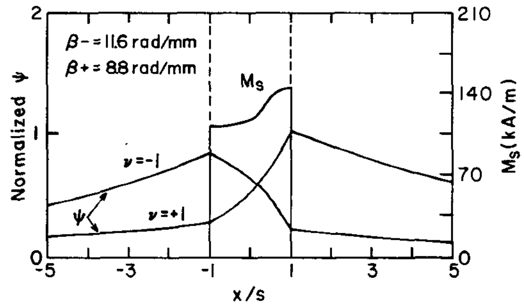


Fig. 2. The MSSW potential for the two directions of propagation in a film with a nonuniform magnetization profile. The principle value of the inverse tangent function was used to describe the thickness variation of $M_s$. The parameters are $f = 2.9$ GHz, $H_0 = 31.8$ kA/m (400 Oe), $2s = 30$ $\mu$m, $t_1 \rightarrow \infty$, and $t_2 = 635$ $\mu$m.

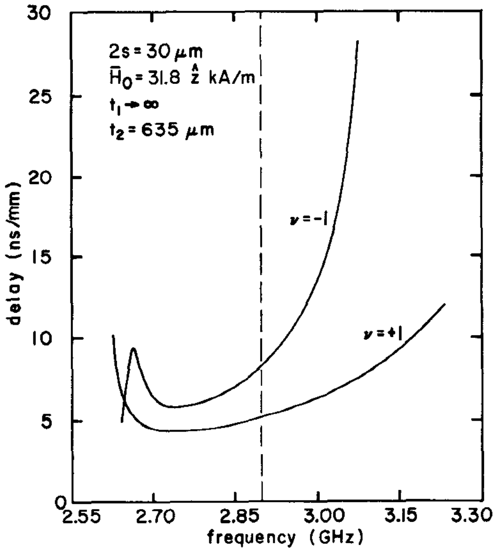


Fig. 3. The delay characteristics of the film shown in Fig. 2. The vertical dashed line indicates the frequency used for the potential profile shown (2.9 GHz).

The MSSW passband for a film of $M_s = 143$ kA/m in a bias field $H_0 = 31.8$ kA/m (400 Oe) extends from $f_L = 2.62$ to $f_H = 3.63$ GHz as given by (26b,c) with $|\gamma| = 28$ GHz/T. The film shown has several other modes for frequencies below $f_L$. The delays shown correspond to the mode associated with the maximum value of $M_s$.

As is well known, the distribution of the MSSW energy depends on the direction of propagation, determined here by $\nu$. The delay peak at $f = 2.67$ GHz in Fig. 2 for $\nu = -1$ is a result of the ground plane. For $\nu = +1$, this effect is no longer apparent because the energy is mostly localized at the right side of the film, far from the ground plane. Instead, the delay turns smoothly upwards near the bottom of the band. We have observed this behavior whenever the maximum value of $M_s$ occurs at the side of the film where most of the energy is concentrated. For the modes shown, the delay near the low-frequency band edge depends primarily on the maximum value of the magnetization. On the other hand, the greatest effect of the ground plane occurs at frequencies that depend on its distance from the film.

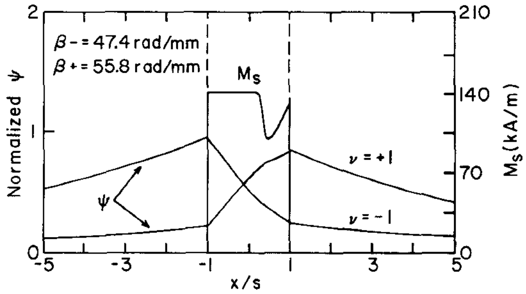


Fig. 4. The MSSW potential for the two directions of propagation in an ion-implanted film at $f = 3$ GHz. Additional parameters are $H_0 = 31.8$ kA/m (400 Oe), $2s = 6$ $\mu$m, $t_1 \rightarrow \infty$, and $t_2 = 635$ $\mu$m.

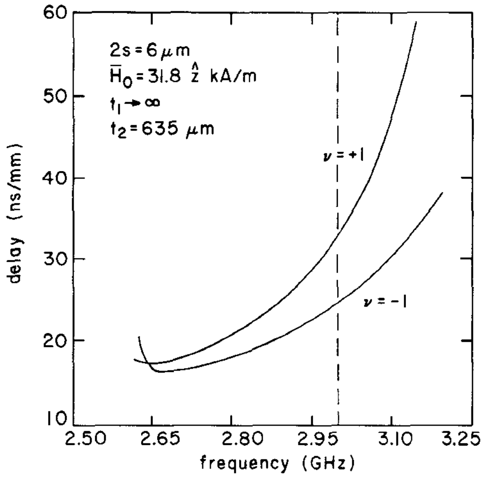


Fig. 5. The delay characteristics of the ion-implanted film shown in Fig. 4. The vertical dashed line indicates the frequency used for the potential profile shown (3 GHz).

A magnetization profile representative of an ion-implanted film [41] is shown in Fig. 4 along with the resulting magnetostatic potentials for both directions of propagation. The corresponding delay characteristics are shown in Fig. 5. The dashed line indicates the frequency used for the calculations of Fig. 4. Here the film thickness is very small in comparison with the ground-plane distance. As a result, the effect of the ground plane is negligible.

In both examples presented above, ten terms were used in the expansion (12b). The absolute error in the wavenumber cannot be calculated as in Section IV since the exact solution is not available. Instead, the convergence of the wavenumber is tested by a modified Cauchy criterion. If $\beta_n$ is the wavenumber found by (19) using $n$ terms in the expansion (12b), then convergence is obtained by requiring the quantity $\zeta_n = |\beta_{n+1} - \beta_n|/\beta_n$ to be sufficiently small. Numerically, this quantity is found to be frequency dependent and largest at the low-frequency band edge. For the calculations of Fig. 3, $\zeta_{10} \leq 10^{-5}$ for $f \geq 2.65$ GHz; for Fig. 5, $\zeta_{10} \leq 0.00135$ for $f \geq 2.64$ GHz.

Previously, the analysis of ion-implanted films was limited to step profiles approximated by multiple implantation [27], [28]. In contrast, *any profile* realizable by multiple or single ion-implantation processes can be readily analyzed with the present method.

## VIII. Conclusions

Magnetization inhomogeneities in ferrite film geometries have been used to control dispersion, form array reflectors, and occur naturally at the film-substrate interface. We have presented a method for analyzing magnetostatic surface-wave modes in thin films with arbitrary variations of $M_s$ through the thickness. Our discussion has been limited to the lowest order modes of the system.

## Appendix A
## Equivalence of Variational Formulation

The equivalence of the variational approach can be demonstrated by taking the variation of $W$ with respect to the potential and its derivatives

$$\delta W = \int_V \delta L \, dV$$

$$\delta W = \int_V \left[ \frac{\partial L}{\partial \psi_k} \delta \psi_k + \frac{\partial L}{\partial \psi_k^*} \delta \psi_k^* \right] dV. \tag{A1}$$

Using integration by parts, (A1) can be rewritten as

$$\delta W = \oint_S \frac{\partial L}{\partial \psi_k} \delta \psi \, d\sigma_k + \oint_S \frac{\partial L}{\partial \psi_k^*} \delta \psi^* \, d\sigma_k - \int_V \frac{\partial}{\partial x^k} \left( \frac{\partial L}{\partial \psi_k} \right) \delta \psi \, dV - \int_V \frac{\partial}{\partial x^k} \left( \frac{\partial L}{\partial \psi_k^*} \right) \delta \psi^* \, dV \tag{A2}$$

where $d\sigma_k = n_k ds$ and $n_k$ is the $k$th component of an outward directed unit vector normal to the surface element $ds$. Viewed as a variational problem $\delta W = 0$, (A2) is associated with the following field equations:

$$\frac{\partial}{\partial x^k} (\mu_{ki} \psi_i)^* = 0 \tag{A3a}$$

$$\frac{\partial}{\partial x^k} (\mu_{ki} \psi_i) = 0 \tag{A3b}$$

and the boundary condition integral

$$\oint_S \left[ (\mu_{ki} \psi_i)^* \delta \psi + (\mu_{kj} \psi_j) \delta \psi^* \right] d\sigma_k = 0. \tag{A4}$$

Here we have made use of the fact that $\bar{\bar{\mu}}$ is Hermitian for a lossless medium (cf. (8)). For Dirichlet boundary conditions (2b), $\delta \psi$ is chosen to vanish on $S$, thus satisfying the condition (A4). For homogeneous Neumann boundary conditions, (A4) is required to vanish for all $\delta \psi$ and $\delta \psi^*$, resulting in (2a). Finally, comparison of (1) and (A3) shows the complete equivalence of the boundary value and the variational approach to the magnetostatic field problem.

## Appendix B
## Stationary Value of $W$

The functional $W$ can be written

$$W = -\int_V \bar{b} \cdot \nabla \psi^* \, dV. \tag{B1}$$

Integration by parts gives

$$W = -\oint_S \psi^* \bar{b} \cdot \hat{n} \, dS + \int_V \psi^* \nabla \cdot \bar{b} \, dV. \tag{B2}$$

The first term vanishes because of the boundary condition $\bar{b} \cdot \hat{n} = 0$ on $S$, and the Maxwell equation $\nabla \cdot \bar{b} = 0$ causes the last term to vanish, yielding the desired result $W = 0$.

## Appendix C
## Invariance of $\beta s$ Under Changes of Scale

The dispersion relation (19) obviously does not change if we replace $a_{ik}$ in (18b) by

$$A_{ik} = a_{ik} s. \tag{C1}$$

This gives

$$A_{ik} = B \left[ th\left( B \frac{t_2}{s} \right) f_i f_k|_{-s} + th\left( B \frac{t_1}{s} \right) f_i f_k|_s \right] + s \int_{-s}^{s} (1+\chi) f_i' f_k' \, dx + B^2 \frac{1}{s} \int_{-s}^{s} (1+\chi) f_i f_k \, dx + \nu B \int_{-s}^{s} \kappa (f_i' f_k + f_i f_k') \, dx \tag{C2}$$

where

$$B = \beta s. \tag{C3}$$

Now consider a geometry related to that of Fig. 1 by a change of scale described by the transformation

$$\tilde{x} = \epsilon x, \qquad \epsilon > 0. \tag{C4}$$

Applying this transformation to (C2) gives

$$\tilde{A}_{ik} = \tilde{B} \left[ th\left( \tilde{B} \frac{\tilde{t}_2}{\epsilon s} \right) \tilde{f}_i \tilde{f}_k|_{\tilde{x} = -\epsilon s} + th\left( \tilde{B} \frac{\tilde{t}_1}{\epsilon s} \right) \tilde{f}_i \tilde{f}_k|_{\tilde{x} = \epsilon s} \right] + \epsilon s \int_{-\epsilon s}^{\epsilon s} (1+\tilde{\chi}) \frac{\partial \tilde{f}_i}{\partial \tilde{x}} \frac{\partial \tilde{f}_k}{\partial \tilde{x}} \, d\tilde{x} + \tilde{B}^2 \frac{1}{\epsilon s} \int_{-\epsilon s}^{\epsilon s} (1+\tilde{\chi}) \tilde{f}_i \tilde{f}_k \, d\tilde{x} + \nu \tilde{B} \int_{-\epsilon s}^{\epsilon s} \tilde{\kappa} \left( \frac{\partial \tilde{f}_i}{\partial \tilde{x}} \tilde{f}_k + \tilde{f}_i \frac{\partial \tilde{f}_k}{\partial \tilde{x}} \right) d\tilde{x}. \tag{C5}$$

Here quantities in the transformed system are indicated by a tilde. We can always choose the basis functions $\tilde{f}_i$ such that

$$\tilde{f}_i(\tilde{x}) = f_i(x) \tag{C6}$$

which implies (cf. (C4))

$$\frac{\partial}{\partial \tilde{x}} \tilde{f}_i(\tilde{x}) = \frac{1}{\epsilon} f_i'(x). \tag{C7}$$

Substituting (C6) and (C7) into (C5) gives

$$\begin{aligned}\tilde{A}_{ik} = \tilde{B}\left[th\left(\tilde{B}\frac{t_2}{s}\right)f_i f_k|_{-s} + th\left(\tilde{B}\frac{t_1}{s}\right)f_i f_k|_s\right] \\ + s\int_{-s}^{s}\left[1+\tilde{\chi}(\epsilon x)\right]f_i' f_k'\,dx \\ + \tilde{B}^2\frac{1}{s}\int_{-s}^{s}\left[1+\tilde{\chi}(\epsilon x)\right]f_i f_k\,dx \\ + \nu\tilde{B}\int_{-s}^{s}\tilde{\kappa}(\epsilon x)\left(f_i' f_k + f_i f_k'\right)dx \qquad \text{(C8)}\end{aligned}$$

where we have used $\tilde{t}_1 = \epsilon t_1$, and $\tilde{t}_2 = \epsilon t_2$ according to (C4). If the inhomogeneity profile is also scaled such that

$$\tilde{\chi}(\tilde{x}) \equiv \tilde{\chi}(\epsilon x) = \chi(x) \qquad \text{(C9a)}$$

$$\tilde{\kappa}(\tilde{x}) \equiv \tilde{\kappa}(\epsilon x) = \kappa(x) \qquad \text{(C9b)}$$

then comparison of (C2) and (C8) clearly yields

$$A_{ik}(B) \equiv \tilde{A}_{ik}(\tilde{B}) \qquad \text{(C10)}$$

which, by (19), implies

$$B(\omega) \equiv \tilde{B}(\omega) \qquad \text{(C11)}$$

or, by (C3)

$$\beta(\omega)s \equiv \tilde{\beta}(\omega)\epsilon s. \qquad \text{(C12)}$$

The above derivation is valid for negative $\epsilon$ as well. In this case, (C12) shows that a change in the direction of propagation is also required to preserve the invariance.

In general, the dispersion relation (19) represents several modes with their associated frequency passbands. Equation (C12) shows that corresponding modes given by $\beta$ and $\tilde{\beta}$ have exactly the same frequency passbands, since multiplication by the factor $\epsilon$ does not change the pole or zero locations of a function.

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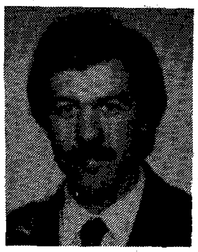

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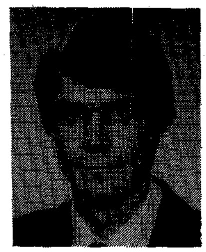

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